

# REFLECTION TIME AND THE GOOS-HÄNCHEN EFFECT FOR REFLECTION BY A SEMI-INFINITE RECTANGULAR BARRIER

Edward R. Floyd  
10 Jamaica Village Road, Coronado, California 92118-3208  
floyd@crash.cts.com

Received 1 November 1996; revised 26 April 2000

Version 3

Final version consistent with galley proofs  
To be published in *Found. Phys. Lett.*

## Abstract

The reflection time, during which a particle is in the classically forbidden region, is described by the trajectory representation for reflection by a semi-infinite rectangular barrier. The Schrödinger wave function has microstates for such reflection. The reflection time is a function of the microstate. For oblique reflection, the Goos-Hänchen displacement is also a function of the microstate. For a square well duct, we develop a proposed test where consistent overdetermination of the trajectory by a redundant set of observed constants of the motion would be beyond the Copenhagen interpretation.

PACS Numbers: 3.65.Bz; 3.65.Ca; 3.65.Nk

Keywords: reflection time, tunnelling time, dwell time, Goos-Hänchen, trajectory representation, microstates.

## 1 INTRODUCTION

Tunneling times, reflection times and dwell times have been well studied for barriers. Regrettably, we still do not have universally accepted answers for these times; even the concepts for these times remain in dispute [1–4]. Part of the problem is that motion in the classically forbidden region inside the barrier is a quantum process that is confounded by the interference between matter reflecting and tunneling.

Herein, we investigate reflection of particles with sub-barrier energy by a semi-infinite rectangular barrier. This simplifies the problem as all particles are reflected only. We use a trajectory representation [5–8] to derive the reflection time. We compare our results with those extrapolated from other investigations of tunneling. We first study reflection time for normal incidence to show that the trajectory method is valid as it produces reflection times that are consistent with other results. We next study the Goos-Hänchen displacement for oblique incidence. We then consider square wells and ducts. The Goos-Hänchen displacements for a square well duct and other constants of the motion present a proposed, strong test for resolving the

“underdetermination” issue [9] whether the trajectory representation and the Copenhagen interpretations may be distinguished. This proposed test hinges on whether an overdetermined set of constants of the motion is consistent in observation with the theoretical redundancy of this set.

This investigation is an application of the trajectory representation. We begin with the generalized Hamilton-Jacobi equation for the trajectory and its solution, a generalized Hamilton’s characteristic function, which is the generator of the motion [7,8]. Application of the trajectory representation to a quantum problem follows a recipe for processing the generalized Hamilton’s characteristic function of quantum mechanics which in practice is similar to the recipe for processing the classical Hamilton’s characteristic function for the analogous problem in classical mechanics. This recipe is standard; in both cases, the Hamilton-Jacobi transformation equations for constant coordinates (often called Jacobi’s theorem) are the equations of motion. Those unfamiliar with the trajectory representation who are interested in the theoretical foundations of the trajectory representation can find a progressive treatment in references 5 through 8. The trajectory representations differs with Bohmian mechanics [10] even though both spring from essentially the same generalized Hamilton-Jacobi equation. These differences have been discussed elsewhere [5,6,11]. In this exposition, we shall note significant differences as they occur.

We have chosen to examine the semi-infinite rectangular barrier because its Schrödinger representation is familiar. Separation of variables applies. Furthermore, exact solutions exist in closed form in terms of elementary functions for both the trajectory and Schrödinger representations.

Microstates of the Schrödinger wave function exist for reflection by a semi-infinite barrier because this barrier induces a nodal singularity in the trajectory representation that is associated with a non-isolated zero in the Schrödinger wave function [8]. We show that the reflection time and the Goos-Hänchen displacement are deterministic. They depend upon the particular microstate of the reflecting particle. A reflection time or a Goos-Hänchen displacement that is a function of the particular microstate is a manifestation that the Schrödinger wave function is not in general an exhaustive description of nonrelativistic phenomenon.

We include in this investigation a study of oblique incidence where there is a Goos-Hänchen displacement between the incident trajectory’s entry point and the reflected trajectory’s exit point in the barrier [12]. We show that the Goos-Hänchen displacement is dependent upon the particular microstate. The Goos-Hänchen displacement is presented as a candidate to sidestep the conceptual difficulties in tunneling with “clocks” [13] and the absence of a time operator in the operator formulation of nonrelativistic quantum mechanics [14]. Therefore, the Goos-Hänchen displacement offers a better test for resolving the underdetermination than reflection time.

Our investigation of oblique incidence also shows that the trajectory representation is not a hydrodynamic representation, such as Bohmian mechanics, for the trajectories are not stream lines as they become imbedded in the surface of constant Hamilton’s characteristic function at the turning point infinitely deep inside the barrier. This allows velocities to increase without bound along the trajectory at the turning point. This also induces the turning points to be cusps.

The reflection time is finite for either normal or oblique incidence while the distance traversed by the trajectory is infinite. Therefore, the velocity along the trajectory must be infinite over a measurable duration.

We also examine a square well and a square well duct which are extensions of the semi-infinite barrier problem. We report that the reflection time as well as the period of libration are determined by the particular microstate. We show that these times are functions of various constant of the motion.

The trajectory representation is causal and not based upon chance [5–8]. The trajectory representation has shown, by certain tunneling through a finite barrier by a particle with sub-barrier energy, that the Born postulate of the Copenhagen interpretation, which attributes a probability amplitude to the Schrödinger wave function, is unnecessary [15]. Any solitary trajectory or microstate is sufficient by itself to specify the Schrödinger wave function [7]. The set of initial conditions that are necessary and sufficient to specify the particular trajectory or microstate are known [6]. There is no need to invoke an ensemble of trajectories to get the Schrödinger wave function or to describe quantum phenomena. The trajectory representation renders predictions for an individual particle viz-a-viz the probability predictions of the Copenhagen representation for an ensemble of particles [15]. While both the trajectory representation and Bohmian mechanics are causal, Bohmian mechanics is still a stochastic theory [16] where an ensemble of trajectories are needed to

describe quantum phenomena.

Without a stochastic requirement, the trajectory representation does not need a wave packet to describe or localize a particle. This simplifies our investigation by allowing us to use a monochromatic particle. This also generalizes our findings by making them independent of the shape of a wave packet.

In this investigation, we apply the Schrödinger representation without the Copenhagen interpretation which assigns a probability amplitude to the Schrödinger wave function. Lest we forget, Schrödinger opposed the Copenhagen interpretation of his wave function.

For completeness, scattering of particles with super-barrier energy from a semi-infinite barrier has been reported elsewhere [11].

In Section 2, we develop the equations of motion for normal incidence, determine the reflection time, and show that it is dependent upon the particular microstate. We develop the corresponding Schrödinger representation as far as it goes to show that it does not discern the dependence due to microstates. In Section 3, we develop the trajectory equations for oblique incidence and determine the Goos-Hänchen displacement. We also demonstrate that the trajectories are generally not wave normals. In Section 4, we develop the reflection time and the libration period for the square well. We show that reflection time and libration period are described by constants of the motion that are beyond the Copenhagen interpretation. In Section 5, we investigate square well ducts. We develop the Goos-Hänchen displacement and libration displacement as constants of the motion for a trajectory in the square well duct. We show that overdetermining the microstate (trajectory) in a duct renders a strong test for resolving underdetermination.

## 2 NORMAL INCIDENCE

**Equation of Motion:** Let us initially consider the reflection of a particle that is normally incident to a semi-infinite rectangular barrier whose potential is given by

$$V(x) = \begin{cases} 0, & x < 0 \\ U, & x \geq 0 \end{cases}$$

where  $U$  is finite positive. For normal incidence, the particle will have motion only in the  $x$ -direction. We choose a sub-barrier energy given by  $E_x = (\hbar k_x)^2/(2m) < U$  where  $E_x$  is the energy for an incident particle,  $k_x$  is the wavenumber in the  $x$ -direction,  $m$  is mass of the particle, and  $\hbar = h/(2\pi)$  where in turn  $h$  is Planck's constant. For normal incidence, we make it explicit by using  $E_x$  that only the  $x$ -dimension contributes to energy.

The trajectory representation is based upon a generalized Hamilton-Jacobi formulation. The time-independent generalized Hamilton-Jacobi equation for quantum mechanics is a phenomenological equation that is given for one-dimensional motion in the  $x$ -direction with normal incidence by [6–8]

$$\frac{(\partial W/\partial x)^2}{2m} + V - E_x = -\frac{\hbar^2}{4m} \langle W; x \rangle \quad (1)$$

where  $\langle W; x \rangle$  is the Schwarzian derivative of  $W$  with respect to  $x$ . The Schwarzian derivative is given by

$$\langle W; x \rangle = \left[ \frac{\partial^3 W/\partial x^3}{\partial W/\partial x} - \frac{3}{2} \left( \frac{\partial^2 W/\partial x^2}{\partial W/\partial x} \right)^2 \right].$$

In Eq. (1),  $W$  is Hamilton's characteristic function,  $\partial W/\partial x$  is the momentum conjugate to  $x$ . The left side of Eq. (1) manifests the classical Hamilton-Jacobi equation while the Schwarzian derivative on the right side manifests the higher order quantum effects. We explicitly note that  $W$  and  $\partial W/\partial x$  are real even in the classically forbidden region inside the barrier ( $x \geq 0$ ). The general solution for  $\partial W/\partial x$  is given by [7,8]

$$\partial W/\partial x = \pm(2m)^{1/2}(a\phi^2 + b\theta^2 + c\phi\theta)^{-1} \quad (2)$$

where  $(a, b, c)$  is a set of real coefficients such that  $a, b > 0$ , and  $(\phi, \theta)$  is a set of normalized independent solutions of the associated time-independent one-dimensional Schrödinger equation,

$$\frac{-\hbar^2}{2m} \frac{d^2\psi}{dx^2} + (V - E_x)\psi = 0.$$

The independent solutions  $(\phi, \theta)$  are normalized so that their Wronskian,  $\mathcal{W}(\phi, \theta) = \phi d\theta/dx - d\phi/dx \theta$ , is scaled to give  $\mathcal{W}^2(\phi, \theta) = 2m/[\hbar^2(ab - c^2/4)] > 0$ . (The nonlinearity of the generalized Hamilton-Jacobi equation induces this normalization upon  $\mathcal{W}$ .) This ensures that  $(a\phi^2 + b\theta^2 + c\phi\theta) > 0$ . Also, the conjugate momentum is not the mechanical momentum, i.e.,  $\partial W/\partial x \neq m\dot{x}$ . We note for completeness that a particular set  $(\phi, \theta)$  of independent solutions of the Schrödinger equation may be chosen by the superposition principle so that the coefficient  $c$  is zero.

The motion in phase space is specified by Eq. (2). This phase-space trajectory is a function of the set of coefficients  $(a, b, c)$ . The  $\pm$  sign in Eq. (2) designates that the motion may be in either  $x$ -direction.

The corresponding solution for the generalized Hamilton's characteristic function,  $W$ , is given by

$$W = \hbar \arctan \left( \frac{b(\theta/\phi) + c/2}{(ab - c^2/4)^{1/2}} \right) + K \quad (3)$$

where  $K$  is an integration constant which we may set to zero herein.

Hamilton's characteristic function is a generator of motion. The equation of motion in the domain  $[x, t]$  is given by the Hamilton-Jacobi transformation equation for a constant coordinate. The procedure simplifies for coordinates whose conjugate momenta are separation constants. For stationarity,  $E_x$  is a separation constant for time for a trajectory with normal incidence. Thus, the equation of motion for the trajectory time,  $t$ , relative to its constant coordinate  $\tau$ , is given as a function of  $x$  by

$$t - \tau = \partial W / \partial E_x \quad (4)$$

where the trajectory for a given energy,  $E_x$ , is a function of a set of coefficients  $(a, b, c)$  and  $\tau$  specifies the epoch. The equation of motion for the trajectory, Eq. (4), and the Bohmian equation of motion differ. Bohmian mechanics asserts that  $\partial W / \partial x$  in Eq. (1) would be the mechanical momentum,  $m\dot{x}$  and the subsequent integration of Eq. (1) would become Bohm's equation of motion [10].

**Microstates:** A particle with normal incidence and with sub-barrier energy,  $E_x$ , has its turning point at  $x = \infty$  where the Schrödinger wave function goes to a non-isolated zero [7,8]. In the generalized Hamilton-Jacobi representation, the barrier induces a nodal singularity at  $x = \infty$  in  $\partial W / \partial x$  [8]. From Eq. (2),  $\partial W / \partial x \rightarrow 0$  as  $x \rightarrow \infty$ , regardless of the values of the coefficients  $a$ ,  $b$  and  $c$ , because at least one independent solution of the set  $(\phi, \theta)$ , in the classically forbidden region inside the barrier, must increase without bound as  $x \rightarrow \infty$ . Each trajectory, which is specified by the particular values of the coefficients  $a$ ,  $b$  and  $c$ , is a particular microstate of the Schrödinger wave function [7,8]. Each microstate, by itself, is sufficient to specify the quantum results of a single event; there is no need to invoke an ensemble of microstates to describe quantum phenomenon [17].

**Reflection:** A set of independent solutions  $(\phi, \theta)$  for our given semi-infinite rectangular barrier can be chosen as

$$\phi = \left( \frac{2m}{\hbar^2 k_x^2 (ab - c^2/4)} \right)^{1/4} \cdot \begin{cases} \cos[k_x x + \arctan(\kappa/k_x)], & x < 0 \\ \frac{\exp(-\kappa x)}{[1 + (\kappa/k_x)^2]^{1/2}}, & x \geq 0. \end{cases} \quad (5)$$

and

$$\theta = \left( \frac{2m}{\hbar^2 k^2 (ab - c^2/4)} \right)^{1/4} \cdot \begin{cases} \sin[k_x x + \arctan(\kappa/k_x)], & x < 0 \\ \frac{\left( \frac{\kappa}{k_x} + \frac{k_x}{\kappa} \right) \exp(\kappa x) + \left( \frac{\kappa}{k_x} - \frac{k_x}{\kappa} \right) \exp(-\kappa x)}{2[1 + (\kappa/k_x)^2]^{1/2}}, & x \geq 0 \end{cases} \quad (6)$$

where  $\kappa = [2m(U - E_x)]^{1/2}/\hbar$ . The corresponding Wronskian obeys  $\mathcal{W}^2(\phi, \theta) = 2m/[\hbar^2(ab - c^2/4)] \geq 0$  as expected.

The conjugate momentum is given by Eq. (2) as

$$\partial W(E_x, a, b, c, x)/\partial x = \pm(2m)^{1/2}[a\phi^2(E_x, x) + b\theta^2(E_x, x) + c\phi(E_x, x)\theta(E_x, x)]^{-1}$$

where  $\phi$  and  $\theta$  are respectively given by Eqs. (5) and (6) and where the dependence of the conjugate momentum upon energy,  $E_x$ , and the set of coefficients  $(a, b, c)$  is made explicit. By Eqs. (2), (5) and (6),  $\partial W/\partial x$  and  $\partial^2 W/\partial x^2$  are continuous across the barrier interface at  $x = 0$ . As expected, we see from Eq. (6) that the  $\exp(\kappa x)$  term in  $\theta$  will induce a nodal singularity in  $\partial W/\partial x$ , Eq. (2), for any allowed values of the coefficients  $a$ ,  $b$  and  $c$  at the trajectory's turning point at  $x = \infty$ .

We define the reflection time,  $t_R$ , herein as the duration that a particle spends in the classically forbidden region inside the barrier, which is given by the round trip time for the particle between the barrier surface at  $x = 0$  and the turning point at  $x = \infty$ . From the equation of motion, Eq. (4), and Eqs. (3), (5) and (6), the reflection time,  $t_R = 2[t(\infty) - t(0)]$ , is given by

$$t_R = 2 \frac{(ab - c^2/4)^{1/2} [1 + (\kappa/k_x)^2]}{a + c(\kappa/k_x) + b(\kappa/k_x)^2} \frac{m}{\hbar \kappa k_x}. \quad (7)$$

The reflection time,  $t_R$ , is dependent upon the particular trajectory or microstate as specified by the coefficients  $a$ ,  $b$  and  $c$ . We find that  $t_R$  is inversely proportional to  $\kappa$  consistent with Barton [18] who found that tunneling time,  $t_T$ , decreased with increasing  $\kappa$  for wave packets tunneling through inverted oscillator barriers.

The reflection time is finite because  $\kappa$  and  $k_x$  are real and  $\mathcal{W}^2 > 0$ . But the trajectory traverses an infinite distance between the interface at  $x = 0$  and the turning point at  $x = \infty$  in a finite duration of time. Hence, the velocity along the trajectory must be infinite for a measurable duration of time. Note that this infinite velocity exists even though  $\lim_{x \rightarrow \infty} \partial W/\partial x \rightarrow 0$  as already noted. (We discuss later an analogous effect in higher dimensions, cf. § 3, ¶ Trajectory Directions and Wave Normals).

As already noted, all trajectories, regardless of the particular values of the coefficients,  $a$ ,  $b$  and  $c$  have their turning point at  $x = \infty$  because the solution,  $\partial W/\partial x$ , to the generalized Hamilton-Jacobi equation, Eq. (1), has a nodal point there. The set of coefficients  $(a, b, c)$  can be specified the set of initial conditions  $(x_o, \dot{x}_o, \ddot{x}_o)$  for a particle at some finite  $x_o$  [7]. Even if we could not prescribe the set of initial conditions  $(x_o, \dot{x}_o, \ddot{x}_o)$  (due perhaps to limits of practicality, but not to any limits of principle for the trajectory representation), then we would have to determine the distribution of the coefficients  $a$ ,  $b$  and  $c$ . Any distribution of the reflection time,  $t_R$ , over an ensemble of different microstates is due to the particular distribution of the coefficients  $a$ ,  $b$  and  $c$  and not due to any adduced distribution of early turning points at finite depths as hypothesized by quantum interpretations that attribute a probability amplitude to  $\psi$ . In general, these distributions will not be the same.

Let us now examine  $t_R$  for a particular case. Let us assume that  $a = b$  and  $c = 0$ . In such case, the reflection time, Eq. (7), becomes

$$t_R = 2m/(\hbar \kappa k_x) = \hbar/[E_x(U - E_x)]^{1/2}.$$

This finding for monochromatic propagation is consistent with findings for tunneling by wave packets. For rectangular barriers, Hauge and Stovneeng [1] and Olkhovsky and Recami [3] have shown for phase times and for Larmor times respectfully that  $t_R = t_T$  where  $t_T$  is tunneling time. For arbitrarily large thick barriers,

Hartman [19] has shown that  $t_T \approx 2m/(\hbar\kappa k_x)$  for spatial Gaussian wave packets, and Fletcher [20] has shown that  $t_T = \hbar/[E_x(U - E_x)]^{1/2}$  for wave packets with temporal exponential leading and trailing tails.

As the trajectory representation is consistent with other work, we have confidence in applying it to less studied situations in § 3–5.

**Schrödinger Representation:** We now investigate the corresponding Schrödinger representation for monochromatic reflection by a semi-infinite rectangular barrier. As all particles are reflected, the Schrödinger wave function for the  $x$ -component can be represented by a real function within an arbitrary phase factor. The Schrödinger wave function,  $\psi$ , can be represented in trigonometric form as [7,8]

$$\begin{aligned}\psi &= \frac{(2m)^{1/4} \cos(W/\hbar)}{[a - c^2/(4b)]^{1/2} (\partial W/\partial x)^{1/2}} \\ &= \frac{(a\phi^2 + b\theta^2 + c\phi\theta)^{1/2}}{[a - c^2/(4b)]^{1/2}} \cos \left[ \arctan \left( \frac{b(\theta/\phi) + c/2}{(ab - c^2/4)^{1/2}} \right) \right] = \phi.\end{aligned}\quad (8)$$

where  $\psi$  and  $d\psi/dx$  are continuous across the barrier interface at  $x = 0$  as described by Eq. (5). While  $\psi = \phi$  where  $\phi$  is independent of the coefficients  $a$ ,  $b$  and  $c$ , the generator of motion,  $W$ , for each particular trajectory (a microstate of  $\psi$ ) is a function of the coefficients  $a$ ,  $b$  and  $c$  is given by Eq. (3). Hence, the Schrödinger wave function,  $\psi$ , is not an exhaustive description of reflection of a particle with sub-barrier energy by a semi-infinite rectangular barrier.

The intermediate steps in Eq. (8) systematically inject the coefficients  $(a, b, c)$  into the amplitude and phase of  $\psi$  so that  $\psi$  still remains independent of these coefficients. Nevertheless,  $(a, b, c)$  determine the microstate of  $\psi$ . Each microstate of  $\psi$  has its distinguishing amplitude and phase modulation determined by the coefficients  $(a, b, c)$ . We note that the Copenhagen interpretation asserts that  $\psi$  would be the exhaustive description of nonrelativistic quantum phenomenon. Therefore, the Copenhagen school would declare the intermediate steps in Eq. (8) to be extraphysical. Again, we apply the Schrödinger representation herein without the Copenhagen interpretation.

We can represent  $\psi$  in an exponential format as an alternative to Eq. (8) by

$$\psi = \frac{(a\phi^2 + b\theta^2 + c\phi\theta)^{1/2}}{2[a - c^2/(4b)]^{1/2}} \left\{ \exp \left[ i \arctan \left( \frac{b(\theta/\phi) + c/2}{(ab - c^2/4)^{1/2}} \right) \right] + \exp \left[ -i \arctan \left( \frac{b(\theta/\phi) + c/2}{(ab - c^2/4)^{1/2}} \right) \right] \right\}$$

where the first and second exponential terms represent respectfully the incident and reflected waves. Although  $\psi$  in its trigonometric representation, Eq. (8), is independent of the coefficients  $a$ ,  $b$  and  $c$ , this is not the case for exponential case when the incident and reflected waves are considered individually. Neither the incident wave nor the reflected wave, when considered separately, are independent of these coefficients for [8,11]

$$\frac{(a\phi^2 + b\theta^2 + c\phi\theta)^{1/2}}{2[a - c^2/(4b)]^{1/2}} \exp \left[ \pm i \arctan \left( \frac{b(\theta/\phi) + c/2}{(ab - c^2/4)^{1/2}} \right) \right] = \frac{[1 \pm ic/(4ab - c^2)^{1/2}]}{2} \phi \pm \frac{ib}{2(ab - c^2/4)^{1/2}} \theta. \quad (9)$$

When  $a = b$  and  $c = 0$  as in the case studied by Hauge and Stovneng [1] and Olkhovsky and Recami [3], then in the classically allowed region,  $x < 0$ , the generator of motion simplifies to

$$W(E_x, a, b, c, x) \Big|_{a=b, c=0, x<0} = \hbar \arctan \{ \tan[k_x x + \arctan(\kappa/k_x)] \}.$$

This generator of the motion by Eq. (8) is, within a constant phase factor, consistent with an incident plane wave given by

$$\frac{(2m)^{1/4} (k_x - i\kappa)}{2[\hbar k_x (k_x^2 + \kappa^2)]^{1/2}} \exp(ik_x x),$$

and a reflected plane wave given by

$$\frac{(2m)^{1/4}(k_x + i\kappa)}{2[\hbar k_x(k_x^2 + \kappa^2)]^{1/2}} \exp(-ik_x x).$$

This is the particular microstate, where  $a = b$ , and  $c = 0$ , that contemporary physics tacitly assumes when working in the Schrödinger representation [19,20]. This particular microstate manifests rectilinear propagation for both the incident and reflected unmodulated plane waves. The incident and reflected waves for a more general microstate, where  $a \neq b$  or  $c \neq 0$ , have compound modulation in both amplitude and wavenumber by Eq. (9) [11,15]. These waves with compound modulation are still solutions to the Schrödinger equation [11].

**Underdetermination:** As the Copenhagen interpretation does not entertain microstates of  $\psi$ , the Copenhagen predictions for reflection time,  $t_R$ , will differ with the trajectory predictions of reflection time, Eq. (7), which are dependent upon the microstate for the trajectory  $t_R$  is a function of the coefficients  $(a, b, c)$ .

**Bohmian Mechanics:** As the  $x$ -component of the wave function is real within a constant phase factor, Bohmian mechanics asserts that  $W$  would be independent of  $x$  [10]. Therefore, Bohmian mechanics concludes that the particle would stand still at its initial position, inside or outside the barrier or at the barrier interface, and particle reflection would not occur.

### 3 OBLIQUE INCIDENCE

**Trajectory Equation:** Let us now examine the more general case of oblique incidence. We can always choose our cartesian coordinate system so that the trajectory lies in the  $x, y$ -plane. The generalized Hamilton-Jacobi equation and the Schrödinger equation are separable in cartesian coordinates. In a Hamilton-Jacobi representation, the cartesian coordinate  $y$  is cyclic with a transformed constant  $y$ -momentum given by  $\hbar k_y$ . (This prescribes that  $y$ -motion is constant or the  $y$ -component in the Schrödinger representation is an unmodulated wave.) The cartesian coordinate  $z$  is also cyclic but the transformed constant  $z$ -momentum is zero for our choice of orientation for the coordinate system. The generalized Hamilton-Jacobi equation for oblique incidence is given by

$$\frac{(\partial W/\partial x)^2 + (\partial W/\partial y)^2}{2m} + V - E = -\frac{\hbar^2}{4m} \langle W; x \rangle$$

where  $E$  is the energy for a particle with oblique incidence. As  $y$ -momentum is constant, any contribution due to a latent  $\langle W; y \rangle$  is zero. The solution for the generator of the motion is given by

$$W = \hbar \arctan \left( \frac{b(\theta/\phi) + c/2}{(ab - c^2/4)^{1/2}} \right) + \hbar k_y y. \quad (10)$$

In the above,  $\phi$  and  $\theta$  are still given by Eqs. (5) and (6) respectfully, but where now

$$k_x = \left( \frac{2mE}{\hbar^2} - k_y^2 \right)^{1/2}$$

and

$$\kappa = \left( \frac{2m(U - E)}{\hbar^2} + k_y^2 \right)^{1/2}.$$

where  $E$  is constrained so that  $k_x$  and  $\kappa$  are real. Equation (2) is still valid for  $\partial W/\partial x$ .

The trajectory equation, which describes the progress of the trajectory along the cyclic coordinate  $y$  as a function of  $x$ , is the equation of motion produced by the Hamilton-Jacobi transformation equation for the constant (reference) coordinate  $y_o$  given by

$$y - y_o = -\partial W / \partial(\hbar k_y). \quad (11)$$

**Goos-Hänchen Displacement:** The lateral displacement,  $\Delta y$ , between where the trajectory enters and exits the barrier is known as the Goos-Hänchen displacement [12]. By the trajectory equation, Eq. (11), the Goos-Hänchen displacement,  $\Delta y = 2[y(\infty) - y(0)]$ , is given by

$$\Delta y = 2 \frac{(ab - c^2/4)^{1/2} [1 + (\kappa/k_x)^2]}{a + c(\kappa/k_x) + b(\kappa/k_x)^2} \frac{k_y}{\kappa k_x}.$$

Hence, the Goos-Hänchen displacement, for a given energy  $E$  and a given  $k_y$ , is a function of the coefficients  $a$ ,  $b$  and  $c$ . Analogous to the time of reflection,  $\Delta y$  is inversely proportional to  $\kappa$  and  $k_x$ . The Goos-Hänchen effect gives us an alternative to reflection time for describing the quantum effects of reflection.

**Trajectory Directions and Wave Normals:** We already know that the turning point for the trajectory is at  $x = \infty$  inside the semi-infinite barrier. We now investigate the behavior of the trajectory near the turning point by studying Eq. (11) there. As the trajectory approaches the turning point, we have that  $\partial W / \partial x$  goes to zero as

$$\lim_{x \rightarrow \infty} \frac{\partial W}{\partial(\hbar k_y)} \longrightarrow 4(ab - c^2/4)^{1/2} \frac{k_y}{k_x} \frac{x \exp(-2\kappa x)}{1 + (\kappa/k_x)^2}. \quad (12)$$

Thus, as  $x \rightarrow \infty$ , the trajectory becomes asymptotic to the  $x$ -direction regardless of the values of the coefficients  $a$ ,  $b$  and  $c$ . Nevertheless, how  $\partial W / \partial x$  goes to zero is still a function of the coefficients  $a$ ,  $b$  and  $c$ . Hence, the turning point at  $x = \infty$  for the trajectory is a symmetric cusp in the  $x, y$ -plane for all allowed values of  $a$ ,  $b$  and  $c$ . This is just another manifestation that  $\partial W / \partial x$  has a nodal singularity for a particle with sub-barrier energy reflecting from a semi-infinite barrier. A symmetric cusp for a turning point at infinity, which is formed by its two branches with mirror symmetry of each other that approach a common asymptote, is consistent with traversing an infinite displacement in  $x$  while transversing only a finite distance  $y$ . It is also consistent with a Schrödinger wave function that decreases exponentially in the classically forbidden region.

Meanwhile, the wave normal is given by  $\nabla W$ . From Eq. (10), the wave normal is

$$\nabla W = (2m)^{1/2} (a\phi^2 + b\theta^2 + c\phi\theta)^{-1} \hat{i} + \hbar k_y \hat{j}. \quad (13)$$

At the turning point, we know that  $\lim_{x \rightarrow \infty} (a\phi^2 + b\theta^2 + c\phi\theta)^{-1} = 0$  because at least one solution of the set of independent solutions  $(\phi, \theta)$  must increase without bound at the turning point at  $x = \infty$ . Thus, we conclude from Eqs. (12) and (13) that the trajectory's direction at the turning point is embedded in a surface of constant  $W$  because the wave normal is orthogonal to the trajectory. This is a manifestation that the trajectories are not stream lines of a hydrodynamic representation of quantum mechanics. We also conclude that, when the trajectory direction is embedded in a surface of constant  $W$ , its velocity increases without bound (cf. § 2, ¶ Reflection).

**Underdetermination:** The Goos-Hänchen displacement,  $\Delta y$ , renders a better case than  $t_R$  to resolve the underdetermination issue because the Schrödinger representation has an operator for  $y$  but does not have one for time  $t$ . Thus, both the trajectory representation and the Copenhagen have the means to determine  $\Delta y$ . Again as the Copenhagen interpretation does not entertain microstates of  $\psi$ , the Copenhagen predictions for Goos-Hänchen displacement,  $\Delta y$ , will differ with the trajectory predictions of Goos-Hänchen displacement, Eq. (7), which are dependent upon the microstate for the trajectory  $\Delta y$  is a function of the set of coefficients  $(a, b, c)$ .



**Bohmian Mechanics:** As the  $x$ -component of the wave function is real within a constant phase factor, Bohmian mechanics asserts that  $W$  would be independent of  $x$  [10]. Therefore, Bohmian mechanics concludes that the particle would travel in the  $y$ -direction only and would never reflect off the barrier.

## 4 SQUARE WELLS

Let us now investigate reflection inside square wells as an extension of reflection from semi-infinite rectangular barriers. The potential for a square well may be given by

$$V(x) = \begin{cases} U, & |x| > q \\ 0, & |x| \leq q \end{cases} \quad (14)$$

where  $q$  is finite positive and where  $E_x$ ,  $U$ ,  $k_x$  and  $\kappa$  maintain their previous definitions. A finite square well always has at least one symmetric bound state. While our results, Eqs. (17) and (18), are valid for both symmetric and antisymmetric bound states, we discuss only the symmetric bound states to shorten the presentation. We could have arbitrarily chosen to present only the antisymmetric (or odd) bound states just as easily except that not all square wells have sufficient size, that is  $(2mU)^{1/2}q/\hbar > \pi/2$ , to ensure that an antisymmetric bound state exists. For bound states,  $E_x < U$ . Energy is quantized for symmetric bound states by the familiar transcendental equation  $\tan(k_x q) = \kappa/k_x$ , which can be established from either the quantization of the action variable,  $J$ , for symmetric states by [17]

$$J = \oint \partial W / \partial x dx = (2n + 1)h, \quad n = 0, 1, 2, \dots \text{ and } n\pi < (2mU)^{1/2}q/\hbar$$

or the quantization energy,  $E_x$ , so that  $\psi|_{x=\pm\infty} = 0$ . In either case, the quantization is independent of the coefficients  $a$ ,  $b$  and  $c$  [7]. The set of independent solutions  $(\phi, \theta)$  for this square well is chosen such that  $\phi$  represents the symmetric bound state given by

$$\phi = \left( \frac{2m}{\hbar^2 k_x^2 (ab - c^2/4)} \right)^{1/4} \cdot \begin{cases} \cos(k_x q) \exp[-\kappa(x - q)], & x > q \\ \cos(k_x x), & -q \leq x \leq q \\ \cos(k_x q) \exp[\kappa(x + q)] & x < -q. \end{cases} \quad (15)$$

The other solution,  $\theta$ , is unbound and is not unique as any amount of  $\phi$  may be added to it. While  $\phi$  is symmetric for the symmetric bound state, the corresponding  $\theta$  that we have chosen is antisymmetric. We present this unbound solution as

$$\theta = \left( \frac{2m}{\hbar^2 k_x^2 (ab - c^2/4)} \right)^{1/4} \cdot \begin{cases} \frac{\exp[\kappa(x - q)] - \cos(2k_x q) \exp[-\kappa(x + a)]}{2 \sin(k_x q)}, & x > q \\ \sin(k_x x), & -q \leq x \leq q \\ \frac{\cos(2k_x q) \exp[\kappa(x + q)] - \exp[-\kappa(x + q)]}{2 \sin(k_x q)}, & x < -q. \end{cases} \quad (16)$$

The corresponding Wronskian obeys  $\mathcal{W}^2(\phi, \theta) = 2m/[\hbar^2(ab - c^2/4)] \geq 0$  as expected. For bound states, microstates of the Schrödinger wave function exist where the particular choice of the set of coefficients  $(a, b, c)$  specifies a unique trajectory in phase space for a given quantized energy  $E_x$  [8].

By Eqs. (3), (4), (15) and (16) and by the quantizing condition  $\tan(k_x q) = \kappa/k_x$ , we can evaluate the reflection time,  $t_{\pm R}$ , as the time for the round trip between the barrier interface at  $x = \pm q$  and the turning point at  $x = \pm\infty$ . The reflection time,  $t_{\pm R} = 2|t(\pm q) - t(\pm\infty)|$ , is finite for traversing an infinite distance as given by

$$t_{\pm R} = 2 \frac{(ab - c^2/4)^{1/2} [1 + (\kappa/k_x)^2]}{a \pm c(\kappa/k_x) + b(\kappa/k_x)^2} \frac{m}{\hbar \kappa k_x} \quad (17)$$

where the sign for the coefficient  $c$  in the denominator is dependent upon which interface,  $x = \pm q$ , is applicable. The trajectory for the microstate will not be symmetric for  $c \neq 0$  for our choice of  $(\phi, \theta)$ . The existence of unsymmetric microstates of symmetric Schrödinger wave functions has already been discussed elsewhere [5]. Otherwise, Eqs. (7) and (17) are very similar. We note that  $t_{+R}$  and  $t_{-R}$  are constant of the motion.

For completeness, we present the trajectory's period for a complete libration cycle for a bound microstate in our square well. This libration period,  $t_{\text{Libration}} = 2|t(\infty) - t(-\infty)|$ , for a microstate is given from Eqs. (3), (4), (15) and (16) and the quantizing condition  $\tan(k_x q) = \kappa/k_x$  by

$$t_{\text{Libration}} = 4 \frac{(ab - c^2/4)^{1/2} [1 + (\kappa/k_x)^2] [a + b(\kappa/k_x)^2]}{a^2 + (2ab - c^2)(\kappa/k_x)^2 + b^2(\kappa/k_x)^4} \frac{m(q + \kappa^{-1})}{\hbar k_x}. \quad (18)$$

So the libration period is a function of the coefficients  $a$ ,  $b$  and  $c$  even though, as shown elsewhere [7], the action variable and energy are not. For any allowed set of coefficients  $(a, b, c)$ ,  $t_{\text{Libration}}$  is always finite. We note that  $t_{\text{Libration}}$  is another constant of the motion.

In Bohmian mechanics, the one-dimensional bound state is the archetype for a particle standing still in its original position [10].

We defer further comments on underdetermination until our investigation of square well ducts.

## 5 SQUARE WELL DUCTS:

Let us consider a duct whose axis in two dimension  $(x, y)$  is aligned along the  $y$ -axis. The potential,  $V(x)$ , that forms the duct is still the square well potential given by Eq. (14). The cartesian coordinate  $y$  is cyclic with a transformed constant  $y$ -momentum given by  $\hbar k_y$  as was the case oblique incidence given in § 3. The generator of the motion and the equation of motion are given by Eqs. (10) and (11) respectively where the potential is given by Eq. (14). The Goos-Hänchen displacement,  $\Delta y_{\pm R} = 2|y(\pm q) - y(\pm \infty)|$ , is given by

$$\Delta y_{\pm R} = 2 \frac{(ab - c^2/4)^{1/2} [1 + (\kappa/k_x)^2]}{a \pm c(\kappa/k_x) + b(\kappa/k_x)^2} \frac{k_y}{\kappa k_x} \quad (19)$$

where the sign for the subscript of  $\Delta y$  and the coefficient  $c$  in the denominator is dependent upon which interface,  $x = \pm q$ , is applicable.

The corresponding distance transversed in  $y$  by the trajectory during a libration period,  $\Delta y_{\text{Libration}} = 2|y(\infty) - y(-\infty)|$ , for a microstate is given from Eqs. (3), (4), (15) and (16) and the quantizing condition  $\tan(k_x q) = \kappa/k_x$  by

$$\Delta y_{\text{Libration}} = 4 \frac{(ab - c^2/4)^{1/2} [1 + (\kappa/k_x)^2] [a + b(\kappa/k_x)^2]}{a^2 + (2ab - c^2)(\kappa/k_x)^2 + b^2(\kappa/k_x)^4} \frac{k_y(q + \kappa^{-1})}{k_x}. \quad (20)$$

We now know how the trajectory (microstate) in a duct behaves according to its set of coefficients  $(a, b, c)$ . In order to resolve underdetermination, we shall now develop a proposed test whether an observed overdetermined set of constants of the motion, that is accessible to the Copenhagen interpretation, is consistent with theoretical redundancy.

While we have been able for a given  $E_x$  or  $J$  to describe the set of coefficients  $(a, b, c)$  in terms of the set of initial conditions  $[x_o, \dot{x}_o, \ddot{x}_o]$  in the spirit of Hamilton, the Copenhagen school would argue that we cannot measure these initial conditions simultaneously due to the Heisenberg uncertainty principle (but we could identify either  $x$  or  $\dot{x}$ ; note that the Heisenberg uncertainty principle does not address  $\ddot{x}$ ). The Copenhagen school would conclude that we can specify neither the coefficients  $a$ ,  $b$  and  $c$  nor subsequently the microstate, which is consistent with Copenhagen postulate that  $\psi$  is an exhaustive description of nonrelativistic quantum phenomena. In the following three paragraphs, the set of coefficients  $(a, b, c)$  shall be specified by another set of constants of the motion that are measurable by the Copenhagen school.

We can also express the set of coefficients  $(a, b, c)$  in terms of constants of the motion in the spirit of Jacobi. (Note that the coefficients  $a$ ,  $b$  and  $c$  are, in their own right, constants of the motion.) The constants

of the motion  $E_x$  and  $J$  are independent of  $(a, b, c)$  [7]. Let us survey what other constants of the motion we have that are dependent on the coefficients  $(a, b, c)$ . The Wronskian,  $\mathcal{W}^2(\phi, \theta) = 2m/[\hbar^2(ab - c^2/4)]$ , provides such a constant of the motion. In addition,  $\Delta y_+$ ,  $\Delta y_-$  and  $\Delta y_{\text{Libration}}$  are all such constants of the motion. Also, there exists a constant of the motion,  $I$ , which is an Ermakov invariant established by the Ermakov system formed by the generalized Hamilton-Jacobi equation, Eq. (1), and the Schrödinger equation. This constant of the motion is given for bound states by [17]

$$\begin{aligned} I &= \{W'\psi^2 + (\hbar^2/W')[\psi W''/(2W') + \psi']^2\}/(2m)^{1/2} \\ &= [a - c^2/(4b)]^{-1} = b\hbar^2\mathcal{W}^2/(2m) > 0. \end{aligned}$$

Hence,  $I$  is positive definite for bound states (it is zero for unbound states) [17].

We may, for a given  $E$  or  $J$ , describe the microstate in terms of other constants of the motion instead of the set of coefficients  $(a, b, c)$ . This would remove the Copenhagen school's objection regarding specifying the set of coefficients  $(a, b, c)$  by the set of initial conditions  $[x_o, \dot{x}_o, \ddot{x}_o]$ . Three independent constants of the motion are necessary and sufficient to specify the set of coefficients  $(a, b, c)$ . We have already developed a redundant set of five constants of the motion ( $I, \mathcal{W}, \Delta y_+, \Delta y_-, \Delta y_{\text{Libration}}$ ) that have been expressed herein as functions of the set of coefficients  $(a, b, c)$ . The set  $(b, \mathcal{W}, I)$  is redundant as  $b$  can be specified by

$$b = \frac{2m}{\hbar^2} \frac{I}{\mathcal{W}^2}.$$

The coefficients  $a$  and  $c$  are specified by

$$a = \frac{(2m)^{1/2}}{\hbar\mathcal{W}^2} [1 + (\kappa/k_x)^2] \frac{k_y}{\kappa k_x} \left( \frac{1}{\Delta y_+} + \frac{1}{\Delta y_-} \right) - \frac{2mI}{\hbar\mathcal{W}^2} \left( \frac{\kappa}{k_x} \right)^2$$

and

$$c = \frac{(2m)^{1/2}}{\hbar\mathcal{W}^2} [1 + (\kappa/k_x)^2] \frac{k_y}{\kappa k_x} \left( \frac{1}{\Delta y_+} - \frac{1}{\Delta y_-} \right).$$

The set of Goos-Hänchen displacements ( $\Delta y_+, \Delta y_-$ ) and  $\Delta y_{\text{Libration}}$  are measurable in the Copenhagen interpretation for there is an operator for  $y$  in the Schrödinger representation. The Copenhagen school may still demure with the objection that the set  $(I, \mathcal{W}, a, c)$  is redundant and that we need one more constant of the motion independent of this set to specify the set  $(a, b, c)$  by accessible measurements. The set of initial conditions provides us this other constant. The Copenhagen school must stipulate that we can know either  $x_o$  or  $\dot{x}_o$  by the Heisenberg uncertainty principle and perhaps even also know  $\ddot{x}_o$ . Thus, the set  $(\Delta y_+, \Delta y_-, \xi)$  where  $\xi$  is either  $x_o$  or  $\dot{x}_o$  determines the set  $(a, b, c)$  in a means acceptable to the Copenhagen school.

We can now specify the microstate and consequently the trajectory for a particle in a way acceptable to the Copenhagen school. We still have the constant of the motion,  $\Delta y_{\text{Libration}}$ , which has not been used to specify the set of coefficients  $(a, b, c)$ . With the addition of  $\Delta y_{\text{Libration}}$ , we have overdetermined the set of coefficients  $(a, b, c)$ . We may now propose a test of whether the observed overdetermination of a microstate is consistent with the theoretical redundancy in the set of constants of the motion ( $\Delta y_{\text{Libration}}, \Delta y_+, \Delta y_-, \xi$ ) where  $\xi$  is either  $x_o$  or  $\dot{x}_o$ . This set is accessible to the Copenhagen school. Thus, this redundancy is a strong test for resolving the underdetermination issue. If observation is consistent with the theoretical dependence among the super-sufficient set of constants of the motion, then microstates are so, and the Copenhagen interpretation is in variance.

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